Always Finite Entropy and Lyapunov exponents of two-dimensional cellular automata

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Abstract

Given a new definition for the entropy of a cellular automata acting on a two-dimensional space, we propose an inequality between the entropy of the shift on a two-dimensional lattice and some angular analog of Lyapunov exponents.

1 Introduction

A two-dimensional cellular automaton (CA) is a discrete mathematical idealization of a two-dimensional spacetime physical system. The space, consists of a discrete, infinite two-dimensional lattice with the property that each site can take a finite number of different values. The action of the cellular automaton on this space is the change at each time step of each values of the lattice only taking in consideration the values in a neighbourhood and applying a local rule.

Since this automaton act on a two-dimensional lattice, this map has been used as model in many areas (Physics, Biology ...) (see Wolfram[6]) and seems to be more usefull in pratice than the one-dimensional case. Nevertheless there is very few mathematical results in the two-dimensional case

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except the work of Willson [7], Margara [2] about ergodicity and density of periodic points of linear two-dimensional (CA).

The first cellular automaton defined by Von Neumann and Ulam was two-dimensional one for theoritical self-reproducting biological systems as the well knwon game of life defined by J. Conway.

In [3] Shereshevsky defined the first Lyapunov exponents for one-dimensional cellular automata and establish and inequality with the standart metric entropy. Then he asked for a generalisation of these results in higher dimensions and qualify this extension as a challenging problem. This is the topic of this paper : first defined a natural equivalent of metric entropy which is finite for all two-dimensional (CA) and try to rely this value with a generalisation of (CA) Lyapunov exponents in the two-dimensional case.

Entropy is an isomorphism invariant for dynamical systems see (Denker [1])and can be seen as a measure of the disorder of the dynamical system. It is always finite for a one-dimensional (CA). In the two-dimensional case the metric entropy is in general not finite when the shift metric entropy is positive. M. Shereshevsky postulate that a two-dimensional (CA) have an entropy equal to zero or infinite. Some results shows that for additive one the conjecture is true.

So we propose here a new measure of complexity for two-dimensional (CA) : the Always Finite Entropy (AFE) of a two-dimensional cellular automaton F denote by $h_\mu^{2D}(F)$. In section 3 we show by examples that the (AFE) give equivalent values of entropies of one-dimensional cellular automaton rules embeded in a two-dimensional lattice.

Then like in the one-dimensional case (see Shereshevsky [3], Tisseur [5]) we defined analog of Lyapunov exponent of differential dynamical systems for (CA). These Lyapunov exponents can be seen as the speed of propagation of perturbation in a particular direction. These Lyapunov exponent can be seen more like a generalisation of those defined in the one-dimensional case than a direct transciption of the differential case.

Hence using some generalization of Shannon theorem we obtain an inequality between the always finite entropy of the (CA), the two-dimensional entropy $h_\mu(\sigma_1, \sigma_2)$ and a map which depends on directional Lyapunov exponents :

$$h_\mu^{2D}(F) \leq h_\mu(\sigma_1, \sigma_2) \times \left(\int \frac{\lambda^2(\theta)}{2} d\theta + \sqrt(2) \int \lambda(\theta) d\theta + \frac{\pi}{2} \right)$$

It seems that this result can be generalised in higher dimensions.

2 Definitions

2.1 Symbolics systems and cellular automata

Let A be a finite set or alphabet. Denote by A^{**} the set of all two-dimensional concatenations of letters in A. The finite concatenations are called blocks. The number of letters of a block $u \in A^{**}$ is denoted by $|u|$. The set of infinite lattice $x = (x_{(i,j)})_{(i,j)\in\mathbb{Z}^2}$ is denoted by $A^{\mathbb{Z}^2}$. A point $x \in A^{\mathbb{Z}^2}$ is called a configuration. For each integers s and t and each block u, we call cylinder the set $[u]_{(s,t)} = \{y \in A^{\mathbb{Z}^2} : y_{(s+i,t+j)} = u_{(i,j)}\}$. For $i_1 \le i_2$ and $j_1 \le j_2$ in $\mathbb{Z}$ we denote by $x((i_1, i_2), (j_1, j_2))$ the rectangular block $x_{(i_1,j_1)} \dots x_{(i_1,j_2)};\ x_{(i_1+1,j_1)} \dots x_{(i_1+1,j_2)}; \dots\dots x_{(i_2,j_1)} \dots x_{(i_2,j_2)}$. We endow $A^{\mathbb{Z}^2}$ with the product topology. For this topology $A^{\mathbb{Z}^2}$ is a compact metric space. A metric compatible with this topology can be defined by the distance $d(x, y) = 2^{-t}$ where $t = \min\{s = \sqrt{i^2 + j^2} \mid$ such that $x_{(i,j)} \neq y_{(i,j)}\}$. The horizontal shift $\sigma_1 \colon A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is defined by : $\sigma_1(x) = (x_{(i+1,j})_{(i,j)\in\mathbb{Z}^2}$. The vertical shift $\sigma_2 \colon A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is defined by : $\sigma_1(x) = (x_{(i,j+1)})_{(i,j)\in\mathbb{Z}^2}$.

The dynamical system $(A^{\mathbb{Z}^2}, \sigma_1, \sigma_2)$ is called the full shift. A subshift X is a closed shift-invariant subset X of $A^{\mathbb{Z}^2}$ endowed with the shifts σ_1 and σ_2. It is possible to identify (X, σ_1, σ_2) with the set X.

If $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ are two partitions of $A^{\mathbb{Z}^2}$ denote by $\alpha \vee \beta$ the partition $\{A_i \cap A_j\ i = 1 \dots n,\ j = 1, \dots, m\}$. The metric entropy $h_\mu(T)$ of a transformation T is an isomorphism invariant between two μ-preserving transformations. Put $H_\mu(\alpha) = \sum_{A\in\alpha} \mu(A) \log \mu(A)$ where α is a finite partition of the space. The entropy of the partition α is defined as $h_\mu(\alpha) = \lim_{n\to\infty} 1/n H_\mu(\vee_{i=0}^{n-1} T^{-i}\alpha)$ and the entropy of (X, T, μ) as $\sup_\alpha h_\mu(\alpha)$.

A two-dimensional cellular automaton is a continuous self-map F on $A^{\mathbb{Z}^2}$ commuting with the two-sided shift. We can extend the Curtis-Hedlund-Lyndon theorem [1] an state that for every two-dimensional cellular automaton F there exist an integer r and a block map f from A^{r^2} to A such that:

$F(x)_{(i,j)} = f(x_{i-r,j-r}, \dots, x_{(i,j)}, \dots, x_{(i+r,j+r)})$.

As in the one-dimensional case we call radius of the cellular automaton F the integer r which appears in the definition of the associated local rule.

2.2 Entropy of two-sided shift

Recall that σ_1 and σ_2 are respectively the horizontal and the vertical shift acting on the space $A^{\mathbb{Z}^2}$. Let α_1 be the partition of $A^{\mathbb{Z}^2}$ by the central coordinate $(0,0)$ and $(A_n)_{n\in\mathbb{N}}$ be a sequence of finite sets of $\mathbb{Z}^2$. We denote by $|A_n|$ the number of element of A_n. Define $\alpha_n^{\sigma_1,\sigma_2} = \vee_{(i,j)\in A_n}\sigma_1^{-i}\sigma_2^{-j}\alpha_1$ and $\alpha_n^{\sigma_1,\sigma_2}(x)$ as the element of the partition $\alpha_n^{\sigma_1,\sigma_2}$ which contains the point $x \in A^{\mathbb{Z}^2}$.

In [4] Ornstein and Weiss give an extension of the Shannon McMillan theorem for a class of amenable groups. These results can be used for $\mathbb{Z}^2$ actions if the sequence of the finite partitions verifies some special conditions. First the set A_n must be averaging set, this mean that

$$\lim_{n\to\infty} |\sigma_i(A_n)\Delta A_n|/|A_n| = 0, \quad \text{with i=1 or i=2}$$

and Δ denotes the symmetric difference of two sets.

Then to be special averaging, the sequence $(A_n)_{n\in\mathbb{N}}$ have to satisfy $A_1 \subset A_2 \subset \dots A_n$.

Finaly if A_n is a special averaging sequence, for almost all points $x \in A^{\mathbb{Z}^2}$ (see [4]) one has

$$h_\mu(\sigma_1,\sigma_2,\alpha_1) = \lim_{n\to\infty}\frac{1}{|A_n|} - \log\mu\left(\alpha_n^{\sigma_1,\sigma_2}(x)\right).$$

Remark that $h_\mu(\sigma_1,\sigma_2,\alpha_1) = h_\mu(\sigma_1,\sigma_2)$ because α_1 is a generating partition for σ_1, σ_2.

We are going to use this result in the case of sequences $(A_n)_{n\in\mathbb{N}}$ which are not rectangles (in this case we have the equality by definition) and show that

$$\int_{A^{\mathbb{Z}^2}} \lim_{n\to\infty}\frac{-\log\mu\left(\alpha_n^{\sigma_1,\sigma_2}(x)\right)}{|A_n|}d\mu(x) = h_\mu(\sigma_1,\sigma_2).$$

2.3 Always finite entropy of a two-dimensional CA

Let F be a two-dimensional cellular automata and μ a shift ergodic and F invariant measure. Let α be a finite partition of $A^{\mathbb{Z}^2}$. Let $h_\mu(F,\alpha)$ be the entropy of F with respect of the finite partition α. If $h_\mu(\sigma_1,\sigma_2) > 0$ then in general it exists a sequence of partition α_n such that $h_\mu(F,\alpha_n)$ goes to infinty (see the examples below).

For this reason we propose a new kind of entropy for a two-dimensional cellular automata F, the always finite entropy (AFE) denoted by $h_\mu^{2D}(F)$. For this definition we need to defined a sequence of finite partitions α_n^F.

Let B be a two-dimensional squarred block that is to say a double finite squarred sequence of letters $B = (B_{(i,i)})_{(0\leq i\leq n; 0\leq j\leq n)}$. We denote by $[B]_{(0,0)}^{(n,n)}$ the cylinder which is the set of all the points y such that $y_{(i,j)} = B_{(i,j)}$ with $0 \leq i \leq n$ and $0 \leq j \leq n$. We call n-squarred cylinders blocks the cylinders $[B]_{(0,0)}^{(n,n)}$.

For each positive integer denote by α_n the partition of $A^{\mathbb{Z}^2}$ into $|A|^{n^2}$ n-squarred cylinders blocks $[B]_{(0,0)}^{(n,n)}$.

Then define

$$h_\mu^{2D}(F) = \liminf_{n\to\infty} \frac{1}{n} h_\mu(F, \alpha_n).$$

Later we will see (main theorem) why this entropy is always finite.

Questions 1 *If we only ask to fix a number of $k \times n^2$ coordinates ($k \in \mathbb{N}^*$), in a partition α_n^*, is $\liminf_{p\to\infty} h_\mu(F, \alpha_n^*) = h_\mu^{2D}(F)$?*

Questions 2 *If what case $\left(\frac{1}{n} h_\mu(F, \alpha_n)\right)_{n\in\mathbb{N}}$ is a converging sequence?*

2.4 Directional Lyapunov exponents

An almost line $L_{(i,j)}^\theta$ is a doubly infinite sequence of coordinates $(u_n, v_n)_{n\in\mathbb{Z}}$ with $(u_0, v_0) = (i, j)$. Let $E(x)$ be the integer part of a real x. Consider the continuous space $\mathbb{R}^2$ and a line which pass by the coordinate (i, j) and make an angle θ with the vertical line. This line is the set of points (x, y). Starting from $(x, y) = (i, j)$ and making grow the variable x the succession of value $(E(x), E(y))$ gives the sequence $(u_n, v_n)_{n\in\mathbb{N}}$ of the positive part of the almost line $L_{(i,j)}^\theta$. We obtain the negative coordinate by decreasing the variable x and taking again the sequence $(E(x), E(y))$.

Let $W_{(i,j)}^\theta \subset \mathbb{Z}^2$ be the half plane of the lattice consisting of all the coordinates situated in the halph plane delimited by the almost line $L_{(i,j)}^\theta$ and which contains the central coordinate $(0, 0)$. For all point $x \in A^{\mathbb{Z}^2}$ note $W_{(i,j)}^\theta(x)$ the set of all points y such that $y_{(a,b)} = x_{(a,b)}$ for $(a, b) \in W_{(i,j)}^\theta$.

Let define the propagation map of information : $\Lambda_n^F(\theta)$ during n iterations in the direction θ for a two-dimensional cellular automaton F :

$$\Lambda_n^F(\theta)(x) = \min\{s = \sqrt{i^2 + j^2} | F^k\left(W_{(i,j)}^\theta(x)\right) \subset W_{(0,0)}^\theta(F^k(x)) | 1 \leq k \leq n\}.$$

Put $\Lambda_n^F(\theta) = \max\{\Lambda_n^F(\theta)(x) | x \in A^{\mathbb{Z}^2}\}$. Remark that for each real θ there exist also a couple $(i,j) = G_nF(\theta)$ of positive integers such that $\sqrt{i^2+j^2} = \Lambda_n^F(\theta)$.

Lemma 1 *For all real $\theta \in [0, 2\pi]$ the limit $\lim_{n\to\infty} \frac{\Lambda_n^F(\theta)}{n}$ exits. We denote by $\lambda(\theta)$ these limits.*

Proof

First prove that for each couple of integers m, n and point $x \in A^{\mathbb{Z}^2}$, one has $\Lambda_{m+n}^F(\theta) \leq \Lambda_n^F(\theta) + \Lambda_m^F(\theta)$.

To simplify put $(s,t) = G_nF(\theta)$ and $(u,v) = G_nF(\theta')$, $s \approx \lfloor \Lambda_n^F(\theta)\cos(\theta) \rfloor$ and $t \approx \lfloor \Lambda_n^F(\theta)\sin(\theta) \rfloor$.

From the definition of s,t one has for all point $x \in A^{\mathbb{Z}^2}$ $F^n(W_{(s,t)}^\theta(x)) \subset W_{(0,0)}^\theta(F^n(x))$. Then

$$F^{m+n}(W_{(s,t)}^\theta(x)) \subset F^m(W_{(0,0)}^\theta(F^n(x))) = F^m(W_{(u,v)}^\theta(\sigma_1^u \circ \sigma_2^v \circ F^n(x))).$$

And by definition of u and v we obtain

$$F^m(W_{(u,v)}^\theta(\sigma_1^u \circ \sigma_2^v \circ F^n(x))) \subset W_{(0,0)}^\theta(\sigma_1^u \circ \sigma_2^v \circ F^{n+m}(x)).$$

Then $F^{m+n}(W_{(s+u,t+v)}^\theta(\sigma_1^u \circ \sigma_2^v(x))) \subset W_{(0,0)}^\theta(\sigma_1^u \circ \sigma_2^v \circ F^{m+n}(x))$, this implies that $F^{m+n}(W_{(s+u,t+v)}^\theta(x)) \subset W_{(0,0)}^\theta(F^{m+n}(x))$.

Finaly we can conclude arguing that $\Lambda_{n+m}^F(\theta) \leq \sqrt{(s+u)^2+(t+v)^2} = \sqrt{s^2+t^2} + \sqrt{u^2+v^2} = \Lambda_n^F(\theta) + \Lambda_m^F(\theta)$.

If we fix θ, the numeric sequence $\frac{\Lambda_n^F(\theta)}{n}$ is a subadditive sequence an so converge to $\inf_n \frac{\Lambda_n^F(\theta)}{n}$.

□

2.5 Lyapunov exponent surface

The computation of entropy of $\mathbb{Z}^2$ action using the Shannon-Breiman-Mc-Millan Theorem need to know the number of coordinates fixed for an element of the partition $\alpha_n^{n,F}$ (see section 2.2). This number of elements can be seen as the surface of a polygone on the squarre lattice.

First define the sequence of maps G_n from $[0, 2\pi]$ to $\mathbb{Z}^2$ such that $G_n(\theta) = (\lceil\sqrt{2}n\cos\theta\rceil\ ;\ \lceil\sqrt{2}n\sin\theta\rceil)$ if $\theta \in [0, \pi]$ and $G_n(\theta) = (0,0)$ if $\theta \in]\pi, 2\pi]$. Let T_n be the set of coordinates of the squared lattice in the interior of the set of coordinates $(i,j) = G_n F(\frac{2\pi k}{n}) + G_n(\frac{2\pi k}{n})$, $0 \le k \le n$. Let $T_n = \{(a,b) \in \mathbb{N}^2 | \exists (i,j) = G(\theta)(\theta \in [0, 2\pi]) \, | \sqrt{a^2+b^2} \le \sqrt{i^2+j^2} + G(\theta)\}$.

Remark that we decide arbitrairely to rely the steps in the variations of the angle θ ($\triangle\theta = \frac{2\pi}{n}$) to define the set T_n with the number of iterations of the cellular automaton. Some small changes in the definition of T_n will not change the main result.

Let define a new surface T_n^*. The surface T_n^* is the intersection of all the almost half planes $W^{\frac{2\pi k}{n}}_{G(\frac{2\pi k}{n})+GF(\frac{2\pi k}{n})}$ (defined in section 2.4) with $0 \le k \le n$. We call T_n the Lyapunov exponent surface and T_n^* the surface of common behaviour. As it is difficult to give an expression of the surface T_n^* using the Lyapunov exponents $\lambda(\theta)$ in order to establish an inequality with $h_\mu^{2D}(F)$, we are going to show that the part of T_n^* wich do not belong to T_n became very small in comparaison of T_n when n increase.

Let $T_n^{**} = T_n^* - (T_n^* - T_n \cap T_n^*)$.

Lemma 2 *For each point $x \in A^{\mathbb{Z}}$ one has*

$$\lim_{n\to\infty} \frac{|T_n^{**}|}{|T_n^*|} = 1$$

Proof

Let $|T_n^{**}|$ and $|T_n^*|$ be the respective surface of the sets T_n^{**} and T_n^*. Let $DT_n = |T_n^*| - |T_n^{**}|$, one has $\frac{|T_n^{**}|}{|T_n^*|} = \frac{|T_n^*| - DT_n}{|T_n^*|}$. As the sets T_n^* and T_n^{**} are polygones the difference of the two surfaces is at least a sum of n surfaces of triangles. We decide to bound the surface of these triangles by surfaces of rectangles.

For each $0 \le i \le n$ we consider the rectangle r_i defined by the points p_1 $GF(\frac{\pi}{2} - \frac{2\pi i}{n}) + G(\frac{\pi}{2} - \frac{2\pi i}{n})$; $GF(\frac{\frac{\pi}{2}2 - \pi(i+1)}{n}) + G(\frac{\pi}{2} - \frac{2\pi(i+1)}{n})$, the point p_2 which is the intersection of the almost line $L^{\frac{\pi}{2}-\frac{2\pi}{n}}_{(0,0)}$ and $L^{\frac{2\pi}{n}}_{GF(\frac{2\pi(i+1)}{n})+G(\frac{2\pi(i+1)}{n})}$ and the fourth point which closed the rectangle .

Denote $|r_i|$ the surface of this rectangle. If the point p_2 is at the left of side of p_1 then put $|r_i| = 0$. One has $DT_n \le \sum_{i=+}^{n} |r_i|$. One has $r_i \le l_i \times h_i$ where l_i is the width and h_i the lenght. One has $l_i \le h_i \times \tan(\frac{2\pi}{n})$.

Hence

$r_i = l_i \times h_i \le (h_i)^2 \tan(\frac{2\pi}{n})$ and $h_i = \left(\Lambda_n^F(\frac{2\pi}{n}) + \sqrt{2}n\cos 2\pi i/n\right)\tan(\frac{2\pi}{n})$ so $r_i \le (\Lambda_n^F(\frac{2\pi}{n}) + \sqrt{2}n\cos 2\pi i/n)^2(\tan(\frac{2\pi}{n}))^3$ and when n goes big enought $r_i \le (\Lambda_n^F(\frac{2\pi}{n}) + \sqrt{2}n\cos 2\pi i/n)^2(\frac{2\pi}{n})^3 \approx \frac{Kn^2}{n^3} = \frac{K}{n}$ where K is a constant.

Therefore $DT_n \leq n \times r_i \leq \frac{K}{n} \times n = K$. So $\frac{|T_n^{**}|}{|T_n^*|} \leq \frac{|T_n^*|-K}{|T_n^*|}$ then as we suppose $|T_n^*|$ goes to infinity then $\lim_{n\to\infty} \frac{|T_n^{**}|}{|T_n^*|} = 1$.

□

Let α_1 the partition by the central coordinates defined in the section 2.3 and $\alpha_1(x)$ the element of the partition which contains de point x. Denote by $\alpha_n^{\sigma,T_n^*}(x)$ the element of the partition $\vee_{(i,j)\in T_n^*}\sigma_1^i\sigma_2^j(\alpha_1)$ which contains de point x.

Proposition 1 *For all $0 \leq k \leq n$ and all $x \in A^{\mathbb{Z}^2}$ one has $F^k\left(\alpha_n^{\sigma,T_n^*}(x)\right) \subset \alpha_n(F^k(x))$ and $\alpha_n^F(x) \supset \alpha_n^{\alpha,T_n^*}(x)$.*

Proof

Suppose that there exists $y \in \alpha_n^{T_n^*}(x)$ such that there is some $0 \leq k \leq n$ with

$$F^k(y)\left((0,0);(n,n)\right) \neq F^k(x)\left((0,0);(n,n)\right).$$

We are going to define a sequence $(Z_l(x))_{l\in\mathbb{N}}$ of subsets of $\alpha_n^{T_n^*}(x)$. Let

$$Z_l(x) = \{z \in X | z \in \alpha_n^{T_n^*}(x) \cap_{m=0}^{l} W^{\frac{2\pi m}{n}}_{G_n(\frac{2\pi m}{n})+G_nF(\frac{2\pi m}{n})}(x)\}.$$

Remark that $Z_i(x) \supset Z_{i+1}(x)$ and $Z_n(x) = \{x\}$.

From the supposition there exist $0 \leq l < n$ such that there exist $u \in Z_l(x)$ and $v \in Z_{l+1}(x)$ such that for all $0 \leq k \leq n$ one has $F^k(v)\left((0,0);(n,n)\right) = F^k(x)\left((0,0);(n,n)\right)$ and there exists $0 \leq k \leq n$ such that $F^k(u)\left((0,0);(n,n)\right) \neq F^k(x)\left((0,0);(n,n)\right)$. As $u \in Z_l(x)$ and $v \in Z_{l+1}(x)$ then

$$v \in W^{\frac{2\pi l+1}{n}}_{G_n(\frac{2\pi l+1}{n})+G_nF(\frac{2\pi l+1}{n})}(u).$$

It follows that

$$F^i\left(W^{\frac{2\pi l+1}{n}}_{G_n(\frac{2\pi l+1}{n})+G_nF(\frac{2\pi l+1}{n})}(u)\right) \not\subset W^{\frac{2\pi l+1}{n}}_{G_n(\frac{2\pi l+1}{n})}(F^i(u))$$

and

$$F^i\left(W^{\frac{2\pi l+1}{n}}_{G_nF(\frac{2\pi l+1}{n})}(\sigma^{-G_n(\frac{2\pi l+1}{n})}(u)\right) \not\subset W^{\frac{2\pi l+1}{n}}_{(0,0)}(F^i(\sigma^{-G_n(\frac{2\pi l+1}{n})}(u))$$

then $\Lambda_n^F(\frac{2\pi l+1}{n}) > \sqrt{i^2+j^2}$ where $(i,j) = G_nF(\frac{2\pi l+1}{n})$, which which contradict the hypothesis, so we can conclude.

□

3 Examples of h_μ^{2D} computation and a first inequality

In this section we give different justification of the choice of the definition of the Always Finite Entropy. From this section we denote by X the full shift $A^{\mathbb{Z}^2}$.

3.1 Examples

Using tree examples we are going to show that the (AFE) is a natural extension of the standart entropy apply in the one-dimensional case.

In the three examples we consider the two-dimensional lattices on an alphabet which contains two letters. Let $A = \{0, 1\}$ and consider the space $X = A^{\mathbb{Z}^2}$. We endows the space X with the uniform measure μ. This measure is the unique measure which gives the same weight to all cylinders with the same number of fixed coordinates.

• The first example (F_1) is the horizontal shift named σ_1 in section 2.2 . This (CA) is defined by the rule $[F_1(x)]_{(i,j)} = x_{(i+1,j)}$. Using the definition of metric entropy one has

$$h_\mu(F, \alpha_1) = \int_X - \lim_{n\to\infty} \frac{1}{n} \log \mu(\alpha_1^{n,F_1}(x)) d\mu(x).$$

As μ is the uniform measure and F_1 act on each point x in the same way it follows that $h_\mu(F_1, \alpha_1) = -\lim_{n\to\infty} \frac{1}{n} \log \mu(\alpha_1^{n,F_1}(x))$.

Clearly $\alpha_1^{n,F_1}(x) = x((0,0);(n,0))$ so we have $\mu(\alpha_1^{n,F_1}(x)) = 2^{-n-1}$, then $h_\mu(F_1, \alpha_1) = \lim_{n\to\infty} \frac{1}{n}(n-1)\log(2) = \log(2)$.

Using the same methods we obtains $h_\mu(F_1, \alpha_p) = p\log(2)$ and therefore $h_\mu^{2D}(F_1) = \lim_{n\to\infty} \frac{1}{p} p\log(2) = \log(2)$.

We remark that the value of $h_\mu^{2D}(F_1)$ is the same that the value of the standart entropy of the shift in the one-dimensional case and is equal to the entropy of the two shift action $h_\mu(\sigma_1, \sigma_2)$ on X.

• The second example named F_2 is a (CA) based on an additive rule on the right and left first neigbhours. This (CA) is defined by $[F(x)]_{(i,j)} = x_{(i-1,j)} + x_{(i+1,j)}$ modulo 2. In order to take in account the right and left in an independantly way we need to consider partitions α_n with $n \geq 2$. Clearly one has $\alpha_2^{n,F_2}(x) = x((-n,2);(n+1,0))$ so $\mu(\alpha_2^{n,F_2}(x)) = 2^{-2(2n+2)}$ and in general $\mu(\alpha_p^{n,F_2}(x)) = 2^{-p(2n+2)}$ then $h_\mu^{2D}(F_2) = \lim_{p\to\infty} \frac{1}{p} 2p\log(2) = 2\log(2)$.

Remark that the value of $h_\mu^{2D}(F_2)$ is the same that the entropy of the simple additive (CA) in the one-dimensional case.

• The third example (F_3) is an additive rule with the upper and right first neigbhours. This (CA) is defined by the rule $[F(x)]_{(i,j)} = x_{(i,j+1)} + x_{(i+1,j)}$ modulo 2.

Recall that the set $\alpha_p^{n,F_3}(x)$ is the element of the partition $\vee_{i=0}^n F_3^{-i}\alpha_p$ which contains the point x. First remark that for any $x \in X$ one has $\mu(\alpha_2^{0,F_3}(x)) = 2^{-4}$ because we have to fix 4 coordinates, the squarre delimited by the position $(0,0)$ to $(1,1)$.

Fix $x \in X$ and choose y such that $y_{((0,0);(1,1))} = x_{((0,0);(1,1))}$; $F_3(y)_{((0,0);(1,1))}$ $= F_3(x)_{((0,0);(1,1))}$. In this case we have to fix $y_{(0,2)}$ and $y_{(2,0)}$ but we have the possibility to choose $y_{(1,2)}$ or $y_{(2,1)}$. We can not choose these two coordinates in an independantly way. We deduce that $\mu(\alpha_2^{1,F_3}(x))$=$2 \times 2^{-4} \times 2^{-4} = 2^{-7}$.

More generaly for $\alpha_p(x)$ we have to fix p^2 coordinates, then in $\alpha_p^{1,F}(x)$ we must fix p^2+2p coordinates $(y_{(0,p)} \dots y_{(p-1,p)}$ and $(y_{(p,p-1)} \dots y_{(p,0)})$ but there is two ways of choosing the coordinates in the angle $y_{(p-1,p)}$ and $y_{(p,p-1)}$ so $\mu(\alpha_p^{1,F_3}(x))$=$2 \times 2^{-(p^2+2p)} = 2^{2(p-1)}$.

In order that $F_3^2(y) \in \alpha_p(F^2(x))$ we have to fix $2p+1$ more coordinates in y with again two possibilities which multiply the first two possibilities, so $\mu(\{y|F_3^2(y) \in \alpha_p(F^2(x))\}) = 2^2 \times 2^{-(p^2+2p+1)} = 2^{-(2p-1)}$ and

$$\mu(\{y|F_3^2(y) \in \alpha_p(F^2(x));\ F_3(y);\ F(y) \in \alpha_p(F(x));\ y \in \alpha_p(x)\})$$
$$= 2^{-(p^2)} \times 2^{-(2p-1)} \times 2^{-(2p-1)} = 2^{p^2+2(p-1)}.$$

In general, considering all the configuration such that $F_3^i(y) \in \alpha_p(F_3^i(x))$ $(0 \le i \le n)$, we have to fix $2p+i$ more coordinates (with 2^{i+1} possible choices) in order that $F_3^{i+1}(y) \in \alpha_p\left(F_3^{i+1}(x)\right)$. The set $\alpha_p^{n,F_3}(x)$ is the union of $\pi_{i=1}^n 2^i$ cylinders block with a number of $2p+i$ fixed coordinates. Hence in order to have $F_3^i(y) \in \alpha_p(F_3^i(x))$ with $0 \le i \le n$ we have to fix $p^2 + \sum_{i=0}^{n-1}(2p+i)$ coordinates but there is $\pi_{i=1}^n 2^i$ possible choices of coordinates. Finaly as all the configurations have the same weight for each $x \in X$ and one has $\mu\left(\alpha_p^{n,F_3}(x)\right) = 2^{\left(-\left(p^2+\sum_{i=0}^{n-1}(2p+i)\right)+\sum_{i=1}^n 2^i\right)} = 2^{-(p^2+(2p-1)n)}$.

As in general $\mu(\alpha_p^{n,F_3}(x)) = 2^{-((2p-1)n+p^2)}$ then

$$h_\mu(F_3,\alpha_p) = \lim_{n\to\infty} \frac{(2p-1)n+p^2}{n}\log(2) = (2p-1)\log(2)$$

and $h_\mu^{2D}(F) = \lim_{p\to\infty} \frac{1}{p}(2p-1)\log(2) = 2\log(2)$.

Remark that $h_\mu^{2D}(F_3) = h_\mu^{2D}(F_2)$ which seems natural and coherent.

3.2 Equivalent and upper bound maps of h_μ^{2D}

Here we define two maps which can be related to the always finite entropy. The first one is an upper bound required to establish the main inequality with the Directional Lyapunov Exponent, the second one is an equivalent definition of the (AFE).

Proposition 2 *For each F invariant measure μ one has*

$$h_\mu^{2D}(F) \leq \int_X \liminf_{n\to\infty} \frac{-\log \mu(\alpha_n^{n,F}(x))}{n^2} d\mu(x).$$

Proof

Let p_i be a sequence such that $\lim_{n\to\infty} \frac{1}{p_i} h_\mu(\alpha_{p_i}, F) = h_\mu^{2D}(F)$. Given any real $\epsilon > 0$, there exists an integer I such that if $i \geq I$ then $|h_\mu^{2D}(F) - \frac{1}{p_i} h_\mu(\alpha_{p_i}, F)| \leq \epsilon$.

From the definition of the metric entropy and the dominated convergence theorem , one has

$$h_\mu(\alpha_p, F) = \int_X \lim_{n\to\infty} \frac{-\log(\mu(\alpha_p^{n,F}(x)))}{n} d\mu(x) = \lim_{n\to\infty} \int_X \frac{-\log(\mu(\alpha_p^{n,F}(x)))}{n} d\mu(x).$$

As the sequence

$$\left(\int_X -\log(\mu(\alpha_p^{n,F}(x))) d\mu(x) \right)_{n\in\mathbb{N}} = \left(H(\alpha_p^{n,F}) \right)_{n\in\mathbb{N}}$$

is a subadditive sequence (cf Denker [1]), then $\left(\frac{H(\alpha_p^{n,F})}{n} \right)_{n\in\mathbb{N}}$ converge to $\inf_n \frac{H(\alpha_p^{n,F})}{n} = h_\mu(\alpha_p, F)$. Then for all n and p in $\mathbb{N}$ one has $\frac{1}{p} h_\mu(\alpha_p, F) \leq \int_X \frac{-\log(\mu(\alpha_p^{p,F}(x)))}{p^2} d\mu(x)$ and taking $\epsilon \to 0$ we can conclude.

$\square$

The second proposition give a new seing of the always finite entropy.

Proposition 3 *For any F invariant measure μ one has*

$$h_\mu^{2D}(F) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \int_X \frac{-\log\left(\mu\left(\alpha_{\lceil \epsilon\times n\rceil}^{n,F}(x)\right)\right)}{\lceil \epsilon \times n^2 \rceil} d\mu(x).$$

Proof

Let $(p_i)_{i\in\mathbb{N}}$ be a subsequence such that $\lim \frac{1}{p_i} h_\mu(F, \alpha_{p_i}^{p_i,F}) = h_\mu^{2D}(F)$.

Given any real $\eta > 0$, there exists an integer I such that if $i \geq I$ then $|h_\mu^{2D}(F) - \frac{1}{p_i} h_\mu(F, \alpha_{p_i}^{p_i,F})| \leq \eta/3$.

Then there exist an integer N_1 such that for all $n \geq N_1$ one has

$$\left| \frac{1}{p} h_\mu(\alpha_p, F) - \frac{1}{p} \int_X \frac{-\log\left(\alpha_p^n(x)\right)}{n} d\mu(x) \right| \leq \eta/3.$$

Take $\epsilon < \frac{P}{N_1}$, and define n_i a subsequence such that $\lceil \epsilon n_i \rceil = p_i$. Let $L(\epsilon) = \liminf_{n_i\to\infty} \int_X \frac{-\log\left(\alpha_p^{n_i}(x)\right)}{n_i} d\mu(x)$. Denote by n_j a subsequence of n_i such that $\int_X \frac{-\log\left(\alpha_p^{n_j,F}(x)\right)}{n_j} d\mu(x)$ converge to $L(\epsilon)$. There exist an integer N_2 such that if $n_j \geq N_2$ one has

$$\left| \int_X \frac{-\log\left(\alpha_{\lceil \epsilon n_j \rceil}^{n_j,F}(x)\right)}{\lceil {n_j}^2 \epsilon \rceil} d\mu(x) - L(\epsilon) \right| \leq \eta/3.$$

If $N_1 \geq N_2$ take $N = N_1$ and if $N_1 \leq N_2$ put $P = \lceil \epsilon N_2 \rceil$ in the first condition and $N = N_2$. As the sequence $\int_X \frac{-\log\left(\alpha_p^n(x)\right)}{n} d\mu(x)$ is a decreasing sequence we obtain that for all $\epsilon' \leq \epsilon$ and all $n_j \geq N$

$$\left| h_\mu^{2D}(F) - \int_X \frac{-\log\left(\alpha_{\lceil \epsilon n_j \rceil}^{n_j,F}(x)\right)}{\lceil {n_j}^2 \epsilon \rceil} d\mu(x) \right| \leq \eta.$$

So

$$h_\mu^{2D}(F) = \lim_{\epsilon\to 0} \liminf_{n_j\to\infty} \int_X \frac{-\log\left(\mu\left(\alpha_{\lceil \epsilon \times n_j \rceil}^{n_j,F}(x)\right)\right)}{\lceil \epsilon \times {n_j}^2 \rceil} d\mu(x).$$

As the sequences $\left(\int_X \frac{-\log\left(\alpha_p^n(x)\right)}{n} d\mu(x)\right)$ and $\frac{1}{p_i} h_\mu(\alpha_{p_i}, F)$ are a decreasing sequence then $\int_X \frac{-\log\left(\alpha_{\lceil n\epsilon \rceil}^n(x)\right)}{\lceil n^2\epsilon \rceil} d\mu(x) \leq h_\mu^{2D}(F)$ then we can conclude.

□

Remark 1 *This second relation is not usefull to establish an inequality with the shift entropy and the directional Lyapunov exponents.*

3.3 First inequality

Proposition 4 *For all F μ ergodic two-dimensional (CA), one has*

$$h_\mu^{2D}(F) \leq h_\mu(\sigma_1, \sigma_2) \times (\lambda(0) + \lambda(\pi) + 1) \times (\lambda(\pi/2) + \lambda(3\pi/2) + 1).$$

Proof

Using Proposition 2 we only need to show that

$$\int_X \lim_{n\to\infty} \frac{-\log \mu(\alpha_n^{n,F}(x))}{n^2} d\mu(x)$$
$$\leq (\lambda(0) + \lambda(\pi) + 1) \times (\lambda(\pi/2) + \lambda(3\pi/2) + 1).$$

Let R_n be the rectangular set of couple of integers (i, j) such that $-\Lambda_n^F(3\pi/2) \leq i \leq \Lambda_n^F(\pi/2) + n$ and $\Lambda_n^F(\pi) \leq i \leq \Lambda_n^F(0) + n$.

From the definition of T_n^*, clearly one has $T_n^* \subset R_n$, so using Proposition 1 we obtain

$$\alpha_n^{n,F}(x) \supset (\vee_{(i,j)\in T_n^*} \sigma_1^i \circ \sigma_2^j \alpha_n)(x) \supset (\vee_{(i,j)\in R_n} \sigma_1^i \circ \sigma_2^j \alpha_n)(x).$$

Put $(\vee_{(i,j)\in R_n} \sigma_1^i \circ \sigma_2^j \alpha_n)(x) = \alpha_n^R(x)$ we have

$$h_\mu^{2D}(F) \leq \int_X \lim_{n\to\infty} \frac{-\log \mu\left(\alpha_n^R(x)\right)}{n^2} d\mu(x).$$

So we can write

$$h_\mu^{2D}(F) \leq \int_X \lim_{n\to\infty} \frac{-\log \mu(\alpha_n^R(x))}{|R_n|} \times \frac{|R_n|}{n^2} d\mu(x).$$

From de definition of the two-sided shift one has

$$\int_X \lim_{n\to\infty} \frac{-\log \mu(\alpha_n^R(x))}{|R_n|} d\mu(x) = h_\mu(\sigma_1, \sigma_2, \alpha_1) = h_\mu(\sigma_1, \sigma_2).$$

Hence we have

$$h_\mu^{2D}(F) \leq h_\mu(\sigma_1, \sigma_2) \times \lim_{n\to\infty} \frac{|R_n|}{n^2}.$$

The surface R_n is the surface of a rectangle of width $\Lambda_n^F(0) + \Lambda_n^F(\pi)$ and length $\Lambda_n^F(\pi/2) + \Lambda_n^F(3\pi/2)$. So $|R_n(x)| = (\Lambda_n^F(0) + \Lambda_n^F(\pi) + n) \times (\Lambda_n^F(\pi/2) + \Lambda_n^F(3\pi/2) + n)$.

Then

$$\lim_{n\to\infty}\frac{|R_n|}{n^2} = \left(\frac{\Lambda_n^F(0)}{n}+\frac{\Lambda_n^F(\pi)}{n}+\frac{n}{n}\right)\times\left(\frac{\Lambda_n^F(\pi/2)}{n}+\frac{\Lambda_n^F(3\pi/2)}{n}+\frac{n}{n}\right)$$

$$= (\lambda(0)+\lambda(\pi)+1)\times(\lambda(\pi/2)+\lambda(3\pi/2)+1).$$

So we can conclude. □

4 The continuity of the Lyapunov Exponents

The study of the way the information propagate in a specific direction is interesting in itself. The result in this section permit to establish a better upper bound for the value of the always finite entropy because we now consider the Lyapunov surface which is in general leather than the simple squarre defined in the previous equality.

Lemma 3 *For all $n \in \mathbb{N}^*$ big enought and for all $\theta \in [0, 2\pi]$ there exits $\delta > 0$ such that there exists a positive integer K such that for all $\theta' \leq \delta$ one has*

$$\left|\frac{\Lambda_n^F(\theta+\theta')}{n}-\frac{\Lambda_n^F(\theta)}{n}\right| \leq K\times|\theta'|.$$

Proof

First recall that $\Lambda_n^F(\theta) = \max\{\Lambda_n^F(\theta)(x)|x \in X\}$; clearly we have

$$\Lambda_n^F(\theta) = \max\{s = \sqrt{i^2+j^2}|F^k(x)(a) = F^k(y)(a)|x \in X\ ;\ 0 \leq k \leq n\}$$

where $a = ((-r, r); (0, 0))$.

From de definition of G_nF, if $(i, j) = G_nF(\theta)$, then for each $x \in X$ we obtain $F^k(x)\,((-r, r); (0, 0)) = F^k(y)\,((-r, r); (0, 0))$ where $y \in W^{\theta}_{(i,j)}(x)$.

Remark that there exist $\delta \in [0, 2\pi]$ such that if $0 \leq \theta' \leq \delta$ then $|\tan\theta'| \leq 2\theta')$.

As the information can not propagate more than rn coordinates in n iterations, it follows that $F^k(x)\,((-r, r); (0, 0)) = F^k(y)\,((-r, r); (0, 0))$ if $y \in W^{\theta+\theta'}_{(u;v)}(x)$; $u = i+\lceil(r+1)n\times 2|\theta'|\times\cos\theta\rceil$ and $v = j+\lceil(r+1)n\times 2|\theta'|\times\sin\theta\rceil$.

This implies that $\Lambda_n^F(\theta+\theta') = \sqrt{u^2+v^2} = \sqrt{i^2+j^2}+(r+1)n\times 2|\theta'|$.

Finaly it follows that

$$\left|\frac{\Lambda_n^F(\theta+\theta')}{n}-\frac{\Lambda_n^F(\theta)}{n}\right| \leq 2(r+1)|\theta'|.$$

□

Proposition 5 *For each positive integer n the map* $\frac{\Lambda_n^F(\theta)}{n}$ *are continuous map, moreover the map the sequence* $\frac{\Lambda_n^F(\theta)}{n}$ *converge uniformly to* $\lambda(\theta)$ *which is a continuous map.*

Proof

From Lemma 2 one has for all $n \in \mathbb{N}^*$ and for all for all $\theta \in [0, 2\pi]$ there exits $\delta > 0$ such that for all $\theta' \leq \delta$ one has

$$\left| \frac{\Lambda_n^F(\theta + \theta')}{n} - \frac{\Lambda_n^F(\theta)}{n} \right| \leq K \times \theta'.$$

So these maps are uniformly equicontinuous maps.

As the maps are equibounded by r and are uniformly equicontinuous then we can use the Ascoli-Arzela theorem which told us that that there exists a subsequence n_i such that the sequence $\frac{\Lambda_n^F(\theta)}{n}$ are uniformly convergent. As from Lemma 1 the sequence $\frac{\Lambda_n^F(\theta)}{n}$ converge then we can conclude saying that the maps $\frac{\Lambda_n^F(\theta)}{n}$ converge uniformely to $\lambda(\theta)$.

□

5 The inequality

Theorem 1 *For a shift ergodic measure* μ *of a two-dimensional shift and a two-dimensional cellular automata, we have*
$h_\mu^{2D}(F) \leq h_\mu(\sigma_1, \sigma_2) \times \left(\int_0^{2\pi} \frac{\lambda^2(\theta)}{2} + \sqrt{2} \int_0^{\pi} \lambda(\theta))d\theta + \int_0^{\pi} d\theta \right).$

Proof From Proposition 2 one has

$$h_\mu^{2D}(F) \leq \int_X \liminf_{n\to\infty} \frac{-\log \mu\left(\alpha_n^{n,F}(x)\right)}{n^2} d\mu(x).$$

From Proposition 1 one has $\left(\vee_{(i,j)\in T_n^*} \sigma_1^i \sigma_2^j(\alpha_1)\right)(x) = \alpha_n^{\sigma,T_n^*}(x) \subset \alpha_n^{n,F}(x)$, thus

$$h_\mu^{2D}(F) \leq \int_X \liminf_{n\to\infty} \frac{-\log \mu\left(\left(\vee_{(i,j)\in T_n^*} \sigma_1^i \sigma_2^j(\alpha_1)\right)(x)\right)}{n^2} d\mu(x)$$

and

$$h_\mu^{2D}(F) \leq \int_X \liminf_{n\to\infty} \frac{-\log \mu\left(\alpha_n^{\sigma,T_n^*}(x)\right)}{|T_n^*|} \times \frac{|T_n^*|}{n^2} d\mu(x).$$

Since $(T_n^*)_{n\in\mathbb{N}}$ is a special averaging sequence, we can use the extented Shannon-Breiman-McMillan Theorem (cf Orstein [4]) which implies that for μ-almost all points x one has

$$\lim_{n\to\infty} \frac{-\log \mu\left(\alpha_n^{\sigma,T_n^*}(x)\right)}{|T_n^*|} = h_\mu(\sigma_1,\sigma_2,\alpha_1) = h_\mu(\sigma_1,\sigma_2).$$

Hence one obtains

$$h_\mu^{2D}(F) \leq h_\mu(\sigma_1,\sigma_2) \times \liminf_{n\to\infty} \frac{|T_n^*|}{n^2}.$$

From Lemma 1 one has

$$\liminf_{n\to\infty} \frac{|T_n^*|}{n^2} = \liminf_{n\to\infty} \frac{|T_n^{**}|}{T_n^{**}} \frac{T_n^*}{n^2}$$

$$= \liminf_{n\to\infty} \frac{|T_n^{**}|}{n^2} \leq \liminf_{n\to\infty} \frac{|T_n|}{n^2}.$$

Now we try to evaluate $\liminf_{n\to\infty} \frac{|T_n|}{n^2}$. As $|T_n|$ is a sum of triangles of height $h_i = \Lambda_n^F(\frac{2\pi i}{n})$ and base $b_i = \Lambda_n^F(\frac{2\pi i}{n}) \times \tan(\frac{2\pi}{n})$ when $\lfloor \frac{n}{2} \rfloor \leq i \leq n$. For a real η small enought and n big enought we have

$$\frac{|T_n|}{n^2} = \frac{\frac{2\pi}{n} \times \frac{1}{2}\left(\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left(\Lambda_n^F(\frac{2\pi i}{n}) + \sqrt{2}n\right)^2 + \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} \Lambda_n^2(\frac{2\pi i}{n}\right) + \eta}{n^2}$$

$$= \frac{2\pi}{n}\left(\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left(\frac{1}{2}\left(\frac{\Lambda_n^F(\frac{2\pi i}{n})}{n}\right)^2 + \sqrt(2)\frac{\Lambda_n^F(\frac{2\pi i}{n})}{n} + 1\right) + \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} \frac{1}{2}(\frac{\Lambda_n^F(\frac{2\pi i}{n})}{n})^2 + \frac{\eta}{n}\right).$$

From Proposition 5 the sequence of maps $\frac{\Lambda_n^F}{n}$ converge uniformely to the continuous map λ. This implies that

$$\lim_{n\to\infty} \frac{2\pi}{n} \times \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left(\frac{1}{2}\left(\frac{\Lambda_n^F(\frac{2\pi i}{n})}{n}\right)^2 + \sqrt{2}\frac{\Lambda_n^F(\frac{2\pi i}{n})}{n} + 1\right)$$

$$= \lim_{n\to\infty} \frac{2\pi}{n} \times \sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left(\frac{1}{2}\left(\lambda(\frac{2\pi i}{n})\right)^2 + \sqrt(2)\lambda(\frac{2\pi i}{n}) + 1\right).$$

Therefore using the uniform continuity of the map λ and the Riemman definition of the integral we obtain $\liminf_{n\to\infty} \frac{|T_n(x)|}{n^2} = \lim_{n\to\infty} \frac{|T_n(x)|}{n^2}$ and

$$\lim_{n\to\infty} \frac{|T_n(x)|}{n^2} = \int_{\pi}^{2\pi} \frac{\lambda^2(\theta)}{2} d\theta + \int_0^{\pi} \frac{\lambda^2(\theta)}{2} d\theta + \sqrt(2) \int_0^{\pi} \lambda(\theta) d\theta + \int_0^{\pi} d\theta.$$

Finaly

$$h_{\mu}^{2D}(F) \leq h_{\mu}(\sigma_1, \sigma_2) \times \left(\int_0^{2\pi} \frac{\lambda^2(\theta)}{2} d\theta + \sqrt(2) \int_0^{\pi} \lambda(\theta) d\theta + \int_0^{\pi} d\theta \right).$$

□

Remark 2 *We could obtain a better upper bound if we will be able to estimate the value of* $\liminf_{n\to\infty} \frac{|T_n^*|}{n^2}$ *instead of* $\liminf_{n\to\infty} \frac{|T_n|}{n^2}$.

5.1 Computation of $\lambda(\theta)$

We compare here the two kind of upper bound (Proposition 2 and Theorem 1) in the case of the example F_3 and different extensions of this map.

In order to compute the first upper bound, remark that $\lambda(0) = 1$, $\lambda(\pi) = 0$; $\lambda(\pi/2) = 1$ and $\lambda(3\pi/2) = 0$.

So $\lambda_R^{F_3} = (\lambda(0) + \lambda(\pi) + 1) \times (\lambda(\pi/2) + \lambda(3\pi/2) + 1) = 4$. We obtain the inequality

$$h_{\mu}^{2D}(F) = 2\log(2) \leq 4h_{\mu}(\sigma_1, \sigma_2) = 4\log(2).$$

Now try to evaluate the second upper bound :

$$\lambda_T^{F_3} = \int_0^{2\pi} \frac{(\lambda(\theta))^2}{2} d\theta + \sqrt(2) \int_0^{\pi} \lambda(\theta) d\theta + \int_0^{\pi} d\theta.$$

As no perturbation came from the left or under part of the latttice we only consider the Lyapunov exponents between 0 and $\pi/2$, that it to say we use $S_n = R_n \cap T_n \subset T_n^{**}$ where R_n is the set defined in the proof of Proposition 2. So we have to compute

$$\int_0^{\pi/2} \frac{(\lambda(\theta))^2}{2} d\theta + \sqrt(2) \int_0^{\pi/2} \lambda(\theta) d\theta + \frac{\pi}{2}.$$

If $\theta \in [0, \pi/4]$ then $\lambda(\theta) = \cos(\theta)$ and if $\theta \in [\pi/4, \pi/2]$ then $\lambda(\theta) = \cos(\pi/2 - \theta)$.

Then $\lambda_T^{F_3} = \int_0^{\pi/2} \frac{1+\cos(\theta)}{2} d\theta + \sqrt{2} \int_0^{\pi/4} \cos(\theta) d\theta + \pi/2 = (\frac{\pi}{8} + \frac{1}{4}) + \sqrt{2} + \frac{\pi}{2}$. We have $\lambda_T^{F_3} = 3.628 \leq 4$. More generaly for any integer $k \geq 1$ denote by F'_k a (CA) defined by the local rule $[F'_k(x)]_{(i,j)} = x_{(i+k,j)} + x_{(i,j+k)} \mod 2$. In this case we have $\lambda_R^{F_3} = k^2 + 2k + 1$ and $\lambda_T^{F_3} = k^2(\frac{\pi}{8} + \frac{1}{4}) + \sqrt{2}k + \frac{\pi}{2}$. The difference between the two exponents increase when the value of k.

5.2 Conclusion

Because the value of the standard entropy appears to be zero or not finate for a two-dimensional (CA), the Always Finite Entropy is a map which allows to extend two kinds of results which appear in the one-dimensional case. The first one is the value of metric entropy of a (CA) for additive rules with the uniform measure. The second one is to connect the value of the entropy of the (CA) with the entropy of the shift and the speed of propagation of information (Lyapunov exponents).

Is this version of metric entropy will be useful in more general dynamical systems for which the value of the entropy is not finate.

5.3 Questions

In these following questions F always represent a two-dimensional cellular automaton.

- If F is bijective is it true that $h(F) < \infty$ and $h^{2D}(F) = 0$?
- Is it possible that $h_\mu(F) > 0$ but $h_\mu(F) < \infty$ (Shereshevsky conjecture)?
- Is it possible that $h_\mu^{2D}(F) = 0$ and $h_\mu(F) > 0$?
- Is it possible to define directional exponents which do not depend on the maximum and nevertheless are continuous with the direction θ ?
- Is it possible to find a better upper bound which is equal to zero when the cellular automaton is the identity map?
- In the three examples the value of $h_\mu^{2D}(F)$ depends linearly of the radius r of the local rule. The expression of the upper bound depend on some square of $\lambda(\theta)$. We can wonder if there exist some cellular automaton

with the property that h_μ^{2D} is proportional of the square of the radius of the local rule.

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